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UNSTEADY PROBLEM OF CRACK PROPAGATION IN THE BEAM APPROXIMATION

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Unsteady crack propagation is studied in the beam approximation. The Euler and Timoshenko models of beam theory are used. Crack propagation is described using an energy balance equation.

Key words: *elastic beam, Euler beam, Timoshenko beam, energetic criterion of crack growth, surface energy density, normal mode method, Runge–Kutta method.*

The beam approximation of crack theory is much simpler than the theory of cracks in a three-dimensional elastic medium; nevertheless, it is suitable for studying various problems of crack mechanics. The beam approximation can be used to study the layering of multilayered materials and wedging and to calculate the adhesion strength of lining coatings. The beam approximation of crack theory have been developed by Mikhailov [1] and Slepyan [5], who obtained exact relations at the crack tip and self-similar solutions but unsteady crack propagation under arbitrary loading has not been considered. The approach proposed in the present paper allows one to describe unsteady propagation of a crack with unclosed faces for a symmetric crack under arbitrary symmetric loading and for a crack with one end fixed and the other end free under arbitrary loading. Calculation results for a symmetric crack under a uniformly distributed load are given.

The elastic properties of beams are described using the Euler and Timoshenko beam theories. An energy balance equation taking into account the surface energy released during the formation of new surfaces is used as the crack growth criterion. The problem is solved using the normal mode method: the beam deflection is represented as an expansion in eigenfunctions with unknown coefficients dependent on the variable beam length. For the expansion coefficients and the crack length, a system of nonlinear ordinary differential equations and an inequality, which are solved using the Runge–Kutta method of the fourth order.

We consider an elastic prismatic beam glued to a rigid surface. A beam segment of length $2l_0$ is detached from the surface and lies on it. At the time $t = 0$, a uniformly distributed load P directed upward begins to act on the beam; as a result, the beam moves upward, detaching from the rigid surface (Fig. 1). At the crack ends, the rigid fixing conditions are specified. It is required to determine the motion of the beam and the time dependence of the length of the detached segment. It is assumed that in the glued layer, the surface energy density is lower than that in the beam material (otherwise, the crack will propagate in the beam).

1. Formulation of the Problem for an Euler Beam. The motion of an Euler beam is described by the equation

$$\rho h \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = P, \tag{1.1}$$

where ρ is the density of the beam material; h and w are the beam thickness and deflection; E is Young's modulus; and $I = h^3/12$ is the moment of inertia of the beam cross section. The energy balance equation is written as

$$\int_{-l}^{l} P \frac{\partial w}{\partial t}\, dx = \frac{d}{dt}(T + \Pi) + 4\gamma l' \theta(l'), \tag{1.2}$$

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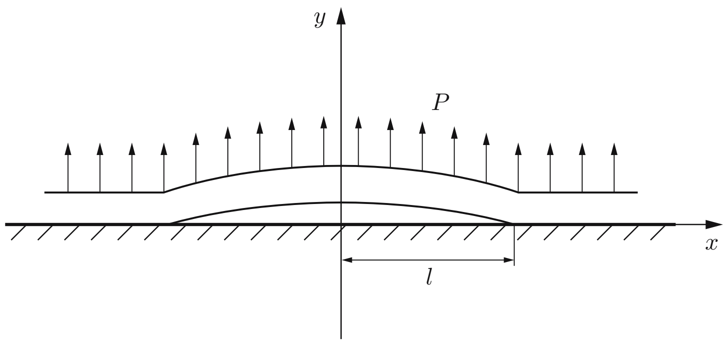

Fig. 1. Diagram of motion of the beam.

where γ is the density of the surface energy released during crack propagation, θ is a Heaviside function, $l = l(t)$ is the crack half-length, and T and Π are the kinetic and potential energies of the beam:

$$T = \frac{\rho h}{2} \int\limits_{-l}^{l} \Big(\frac{\partial w}{\partial t}\Big)^2 dx, \qquad \Pi = \frac{EI}{2} \int\limits_{-l}^{l} \Big(\frac{\partial^2 w}{\partial x^2}\Big)^2 dx.$$

In the last term of Eq. (1.2), the coefficient equal to four takes into account the presence of two cracks at the ends of the beam and the two free surfaces formed during crack propagation. In addition, the inequality

$$l'(t) \geq 0, \tag{1.3}$$

which follows from the physical meaning of the problem (the crack is not closed), should be satisfied. The boundary and initial conditions are written as

$$w(l,t) = w_x(l,t) = 0, \qquad w(-l,t) = w_x(-l,t) = 0; \tag{1.4}$$

$$w(x,0) = w_t(x,0) = 0, \qquad l(0) = l_0, \qquad l'(0) = 0. \tag{1.5}$$

2. Solution of the Problem for an Euler Beam. The beam deflection is represented as

$$w(x,t) = \sum_{k}^{N} A_k(t) W_k(x), \tag{2.1}$$

where $A_k(t)$ are unknown expansion coefficients, $W_k(x)$ are even eigenfunctions for an Euler beam with boundary conditions (1.4), and N is the number of expansion modes. The eigenfunctions $W_k(x)$ satisfy the equation

$$\frac{d^4 W_k}{dx^4} = \lambda_k^4 W_k(x)$$

and boundary conditions (1.4). The even eigenfunctions have the form

$$W_k(x) = \cos(\lambda_k x) - \frac{\cos(\lambda_k l)}{\cosh(\lambda_k l)} \cosh(\lambda_k x),$$

where the eigennumbers $\lambda_k = \lambda_k(l)$ are given by the relation

$$\tan(\lambda_k l) + \tanh(\lambda_k l) = 0. \tag{2.2}$$

As $k \to \infty$, $\tanh(\lambda_k l) \simeq 1$ and $\lambda_k \simeq (k - 1/4)\pi$. We note that the last approximate equality is satisfied fairly well for $k \geq 2$. The eigenfunctions are orthogonal:

$$\int\limits_{-l}^{l} W_k(x) W_m(x)\, dx = B_k \delta_{km}, \qquad B_k = \int\limits_{-l}^{l} W_k^2(x)\, dx \tag{2.3}$$

(δ_{km} is the Kronecker symbol). Relations (2.2) imply that

$$\lambda_k l = \text{const}, \qquad \frac{d\lambda_k}{dl} = -\frac{\lambda_k}{l}, \qquad \frac{\partial}{\partial l} = -\frac{x}{l}\,\frac{\partial}{\partial x}.$$

Then, from (2.1) we obtain

$$\frac{\partial w}{\partial t} = \sum_{k=1}^{N} A_k'(t) W_k(x) - \frac{l'}{l} \sum_{k=1}^{N} A_k(t) x \,\frac{\partial W_k}{\partial x},$$

$$\frac{\partial^2 w}{\partial t^2} = \sum_{k=1}^{N} A_k''(t) W_k(x) - 2\,\frac{l'}{l} \sum_{k=1}^{N} A_k'(t) x \,\frac{\partial W_k}{\partial x} - \Big(\frac{l''}{l} - 2\,\frac{l'^2}{l^2}\Big) \sum_{k=1}^{N} A_k(t) x \,\frac{\partial W_k}{\partial x} + \frac{l'^2}{l^2} \sum_{k=1}^{N} A_k(t) x^2 \,\frac{\partial^2 W_k}{\partial x^2}.$$

The equation of motion for the beam (1.1) becomes

$$\rho h\Big[\sum_{k=1}^{N} A_k''(t) W_k(x) - \Big(\frac{l''}{l} - 2\,\frac{l'^2}{l^2}\Big) \sum_{k=1}^{N} A_k(t) x \,\frac{\partial W_k}{\partial x} - 2\,\frac{l'}{l} \sum_{k=1}^{N} A_k'(t) x \,\frac{\partial W_k}{\partial x}$$

$$+ \frac{l'^2}{l^2} \sum_{k=1}^{N} A_k(t) x^2 \,\frac{\partial^2 W_k}{\partial x^2}\Big] + EI \sum_{k=1}^{N} A_k(t) \lambda_k^4 W_k(x) = P.$$

Multiplying this equation by $W_m(x)$ and integrating the result over x, we obtain

$$\rho h\Big[A_m'' B_m - \Big(\frac{l''}{l} - 2\,\frac{l'^2}{l^2}\Big) \sum_{k=1}^{N} C_{mk} A_k - 2\,\frac{l'}{l} \sum_{k=1}^{N} C_{mk} A_k' + \frac{l'^2}{l^2} \sum_{k=1}^{N} G_{mk} A_k\Big] + EI\lambda_m^4 A_m(t) B_m = PU_m. \tag{2.4}$$

Here

$$C_{mk} = \int_{-l}^{l} x W_m(x)\,\frac{\partial W_k}{\partial x}\,dx, \quad G_{mk} = \int_{-l}^{l} x^2 W_m(x)\,\frac{\partial^2 W_k}{\partial x^2}\,dx, \quad U_m = \int_{-l}^{l} W_m(x)\,dx. \tag{2.5}$$

The kinetic and potential energy are written as

$$T = \frac{\rho h}{2}\Big(\sum_{k=1}^{N} A_k'^2(t) B_k - 2\,\frac{l'}{l} \sum_{k=1}^{N} C_{mk} A_k A_m' + \frac{l'^2}{l^2} \sum_{k=1}^{N} D_{mk} A_k A_m\Big),$$

$$\Pi = \frac{EI}{2} \sum_{k=1}^{N} A_k^2(t) \lambda_k^4 B_k, \qquad D_{mk} = \int_{-l}^{l} x^2\,\frac{\partial W_m}{\partial x}\,\frac{\partial W_k}{\partial x}\,dx.$$

Calculating the integrals, we express the coefficients of the matrices and vectors as

$$G_{mk} = -2C_{mk} - D_{mk}, \quad B_m = l\bar{B}_m, \quad U_m = l\bar{U}_m, \quad C_{mk} = l\bar{C}_{mk}, \quad D_{mk} = l\bar{D}_{mk}, \tag{2.6}$$

where the coefficients $\bar{B}_m$, $\bar{U}_m$, $\bar{C}_{mk}$, $\bar{D}_{mk}$ do not depend on l:

$$\bar{B}_m = \frac{2}{\tanh^2(\lambda_m l) + 1}, \quad \bar{U}_m = 4\,\frac{\sin(\lambda_m l)}{\lambda_m l}, \quad \bar{C}_{mk} = 8\,\frac{\lambda_m^2 \lambda_k^2 \cos(\lambda_m l)\cos(\lambda_k l)}{\lambda_k^4 - \lambda_m^4},$$

$$\bar{D}_{mk} = \frac{8\lambda_k^2\lambda_m^2}{\lambda_m^4 - \lambda_k^4}\Big(\frac{4(\lambda_m^4 + \lambda_k^4)}{\lambda_m^4 - \lambda_k^4}\cos(\lambda_k l)\cos(\lambda_m l) \tag{2.7}$$

$$+ \lambda_m l \sin(\lambda_m l)\cos(\lambda_k l) - \lambda_k l \sin(\lambda_k l)\cos(\lambda_m l)\Big), \quad m \neq k,$$

$$\bar{D}_{kk} = 2\,\frac{(\lambda_k l)^2 \tanh^2(\lambda_k l)/3 - \lambda_k l \tanh(\lambda_k l) + 3/2}{\tanh^2(\lambda_k l) + 1}.$$

In this case, $\bar{C}_{km} = -\bar{C}_{mk}$ for $k \neq m$, $\bar{C}_{kk} = -\bar{B}_k/2$, and $\bar{D}_{mk} = \bar{D}_{km}$.

The energy balance equation (1.2) becomes

$$P\sum_{k=1}^{N} A_k' l\bar{U}_k + l'P\sum_{k=1}^{N} A_k\bar{U}_k = \rho h\Big[\sum_{m=1}^{N} A_m' A_m'' l\bar{B}_m + \frac{l'}{2}\sum_{m=1}^{N} A_m'^2\bar{B}_m - l''\sum_{k,m=1}^{N}\bar{C}_{mk}A_kA_m'$$

$$+\Big(\frac{l'l''}{l}-\frac{l'^3}{2l^2}\Big)\sum_{k,m=1}^{N}\bar{D}_{mk}A_kA_m - l'\sum_{k,m=1}^{N}\bar{C}_{mk}(A_k'A_m' + A_kA_m'') + \frac{l'^2}{l}\sum_{k,m=1}^{N}\bar{D}_{mk}A_kA_m'\Big]$$

$$+ EI\sum_{m=1}^{N} A_mA_m'\lambda_m^4 l\bar{B}_m - 3\,\frac{EIl'}{2l^4}\sum_{m=1}^{N} A_m^2(\lambda_m l)^4\bar{B}_m + 4\gamma l'\theta(l'). \quad (2.8)$$

Multiplying Eq. (2.4) by A_m', summing the result over m, and subtracting the obtained equation from Eq. (2.8), we have

$$P\sum_{k=1}^{N} A_k\bar{U}_k = \rho h\Big[\Big(\frac{l''}{l}-\frac{l'^2}{2l^2}\Big)\sum_{k,m=1}^{N}\bar{D}_{mk}A_kA_m - \sum_{k,m=1}^{N}\bar{C}_{mk}A_kA_m''$$

$$+ 2\,\frac{l'}{l}\sum_{k,m=1}^{N}\bar{D}_{mk}A_kA_m'\Big] - 3\,\frac{EI}{2l^4}\sum_{m=1}^{N} A_m^2(\lambda_m l)^4\bar{B}_m + 4\gamma\theta(l').$$

Substitution of the expression for $A_m''(t)$ into this equation yields

$$\frac{l''}{l}\sum_{k,m=1}^{N} F_{mk}A_kA_m = \frac{P}{\rho h}\sum_{k=1}^{N} A_k\Big(\bar{U}_k + \sum_{m=1}^{N}\frac{\bar{C}_{mk}\bar{U}_m}{\bar{B}_m}\Big) + 3\,\frac{EI}{2\rho h l^4}\sum_{m=1}^{N} A_m^2(\lambda_m l)^4\bar{B}_m$$

$$-\frac{4\gamma\theta(l')}{\rho h} - 2\,\frac{l'}{l}\sum_{k,m=1}^{N} F_{mk}A_kA_m' + \frac{l'^2}{l^2}\sum_{k,m=1}^{N} H_{mk}A_kA_m, \quad (2.9)$$

$$F_{mk} = \bar{D}_{mk} - \sum_{j=1}^{N}\frac{\bar{C}_{jk}\bar{C}_{jm}}{\bar{B}_j}, \qquad H_{mk} = \frac{\bar{D}_{mk}}{2} + \sum_{j=1}^{N}\frac{\bar{C}_{jk}\bar{D}_{jm}}{\bar{B}_j}.$$

The initial conditions (1.5) imply that

$$A_m(0) = A_m'(0) = 0, \qquad l(0) = l_0, \qquad l'(0) = 0. \quad (2.10)$$

Thus, we have Eqs. (2.4) and (2.9) and inequality (1.3) to calculate the motion of the beam. Substituting the expression for l'' from (2.9) into Eq. (2.4), we obtain a system of nonlinear equations of the second order, which is resolved for the higher derivatives. This system needs to be supplemented by inequality (1.3) with initial conditions (2.10). The obtained system reduces to a system of equations of the first order and is solved together with inequality (1.3) using the Runge–Kutta method of the fourth order [6]. Because all terms of Eqs. (2.4) and (2.9) are differentiable and $l(t) \geq l_0$, it follows that in order for the solution to be unique, it is necessary that the inequality $R = \sum_{j=1}^{N} F_{mk}A_mA_k \geq \varepsilon > 0$ [7] be satisfied. The matrix F is symmetric. The eigenvalues of the matrix F were investigated for various numbers of modes. It turned out that the spectrum of the matrix has both positive and negative eigenvalues. Thus, the quantity R can change sign, and, hence, branching of the solutions of this problem is possible. At the initial time, $R = 0$, as follows from the initial conditions. However, at the initial time, the crack does not grow: $l'(0) = 0$; therefore, in the first time step, Eq. (2.4) was solved for fixed crack length. In the next time steps, the quantity $R(t)$ was controlled, and in the calculations, it was positive. The calculation results for different numbers of modes (5, 10, and 20) are in good agreement with each other, which is evidence for the stability of the computational model.

3. Formulation of the Problem for a Timoshenko Beam. The motion of a Timoshenko beam is described by the system [8, 9]

$$\rho h\,\frac{\partial^2 w}{\partial t^2}+hG\Big(\frac{\partial\beta}{\partial x}-\frac{\partial^2 w}{\partial x^2}\Big)=P,\qquad G=\frac{E}{2(1+\nu)},$$

$$\rho I\,\frac{\partial^2\beta}{\partial t^2}+hG\Big(\beta-\frac{\partial w}{\partial x}\Big)-EI\,\frac{\partial^2\beta}{\partial x^2}=0,$$

where G is the shear modulus, ν is Poisson ratio, β is the angle of rotation of the beam cross section ignoring shear, the tangent slope to the beam is given by the relation $\partial w/\partial x=\beta+\psi$, ψ is the shear angle.

In addition, use is made of the energy balance equation (1.2), where

$$T=\frac{1}{2}\int\limits_{-l}^{l}\Big[\rho h\Big(\frac{\partial w}{\partial t}\Big)^2+\rho I\Big(\frac{\partial\beta}{\partial t}\Big)^2\Big]dx,\qquad \Pi=\frac{1}{2}\int\limits_{-l}^{l}\Big[EI\Big(\frac{\partial\beta}{\partial x}\Big)^2+Gh\Big(\frac{\partial w}{\partial x}-\beta\Big)^2\Big]dx,$$

and inequality (1.3). The boundary and initial conditions are written as

$$w(l,t)=\beta(l,t)=0,\qquad w(-l,t)=\beta(-l,t)=0; \tag{3.1}$$

$$w(x,0)=w_t(x,0)=0,\quad \beta(x,0)=\beta_t(x,0)=0,\quad l(0)=l_0,\quad l'(0)=0. \tag{3.2}$$

4. Solution of the Problem for a Timoshenko Beam. As for the Euler beam model, a solution is sought in the form of expansion in eigenfunctions. Eigenfunctions for a Timoshenko beam with boundary conditions similar to (3.1) and (3.2) are found in [9]. We introduce the following notation (see [3]):

$$c_0^2=E/\rho,\qquad c_2^2=G/\rho,\qquad a^2=c_0^2h^2/12.$$

Then, for the eigenfunctions $W_n(x)$ and $\beta_n(x)$, we have the system

$$\begin{aligned}\frac{\partial^2 W_n}{\partial x^2}-\frac{\partial\beta_n}{\partial x}&=-\frac{\omega_n^2}{c_2^2}\,W_n,\\ \frac{\partial^2\beta_n}{\partial x^2}+\frac{c_2^2}{a^2}\Big(\frac{\partial W_n}{\partial x}-\beta_n\Big)&=-\frac{\omega_n^2}{c_0^2}\,\beta_n.\end{aligned} \tag{4.1}$$

Eliminating β_n from this system, we find the following equation for W_n:

$$\frac{\partial^4 W_n}{\partial x^4}+\omega_n^2\Big(\frac{1}{c_2^2}+\frac{1}{c_0^2}\Big)\frac{\partial^2 W_n}{\partial x^2}+\Big(\frac{\omega_n^4}{c_2^2c_0^2}-\frac{\omega_n^2}{a^2}\Big)W_n=0. \tag{4.2}$$

A solution to Eq. (4.2) is sought in the form $W_n(x)=\exp(k_nx)$. Then, the coefficients k_n are defined by the expression

$$k_n^2=-\frac{\omega_n^2}{2}\Big(\frac{1}{c_2^2}+\frac{1}{c_0^2}\Big)\pm\sqrt{\frac{\omega_n^4}{4}\Big(\frac{1}{c_2^2}+\frac{1}{c_0^2}\Big)^2-\frac{\omega_n^4}{c_2^2c_0^2}+\frac{\omega_n^2}{a^2}}.$$

From this, we obtain four roots:

$$k_n^{1,2}=\pm i\lambda_n,\qquad k_n^{3,4}=\begin{cases}\pm\mu_n, & \omega_n^2<12c_2^2/h^2,\\ \pm i\mu_n, & \omega_n^2>12c_2^2/h^2,\end{cases}$$

$$\lambda_n=\sqrt{\frac{\omega_n^2}{2}\Big(\frac{1}{c_2^2}+\frac{1}{c_0^2}\Big)+\sqrt{\frac{\omega_n^4}{4}\Big(\frac{1}{c_2^2}+\frac{1}{c_0^2}\Big)^2-\frac{\omega_n^4}{c_2^2c_0^2}+\frac{\omega_n^2}{a^2}}},$$

$$\mu_n=\sqrt{\Big|\sqrt{\frac{\omega_n^4}{4}\Big(\frac{1}{c_2^2}+\frac{1}{c_0^2}\Big)^2-\frac{\omega_n^4}{c_2^2c_0^2}+\frac{\omega_n^2}{a^2}}-\frac{\omega_n^2}{2}\Big(\frac{1}{c_2^2}+\frac{1}{c_0^2}\Big)\Big|}.$$

Taking into account the symmetry of the problem about the Oy axis, we find the even functions $W_n(x)$ and their corresponding odd functions $\beta_n(x)$. Then, for $\omega_n^2 < 12c_2^2/h^2$, we have the representation

$$W_n(x) = a_n \cos(\lambda_n x) + b_n \cosh(\mu_n x), \qquad \beta_n(x) = c_n \sin(\lambda_n x) + d_n \sinh(\mu_n x), \tag{4.3}$$

and, for $\omega_n^2 > 12c_2^2/h^2$, the representation

$$W_n(x) = a_n \cos(\lambda_n x) + b_n \cos(\mu_n x), \qquad \beta_n(x) = c_n \sin(\lambda_n x) + d_n \sin(\mu_n x). \tag{4.4}$$

Substitution of expressions (4.3) and (4.4) into system (4.1) yields

$$c_n = a_n(\omega_n^2/c_2^2 - \lambda_n^2)/\lambda_n, \qquad d_n = \begin{cases} b_n(\omega_n^2/c_2^2 + \mu_n^2)/\mu_n, & \omega_n^2 < 12c_2^2/h^2, \\ b_n(\omega_n^2/c_2^2 - \mu_n^2)/\mu_n, & \omega_n^2 > 12c_2^2/h^2. \end{cases}$$

The coefficients a_n and b_n and the eigenfrequencies ω_n are determined from boundary conditions (3.1). A nonzero solution of the system is possible only if the determinant of the system vanishes. Then, we obtain

$$b_n = \begin{cases} -a_n \cos(\lambda_n l)/\cosh(\mu_n l), & \omega_n^2 < 12c_2^2/h^2, \\ -a_n \cos(\lambda_n l)/\cos(\mu_n l), & \omega_n^2 > 12c_2^2/h^2. \end{cases}$$

In this case, the eigenfrequencies are given by the equations

$$\frac{\omega_n^2/c_2^2 + \mu_n^2}{\mu_n} \tanh(\mu_n l) - \frac{\omega_n^2/c_2^2 - \lambda_n^2}{\lambda_n} \tan(\lambda_n l) = 0, \qquad \omega_n^2 < \frac{12c_2^2}{h^2}; \tag{4.5}$$

$$\frac{\omega_n^2/c_2^2 - \mu_n^2}{\mu_n} \tan(\mu_n l) - \frac{\omega_n^2/c_2^2 - \lambda_n^2}{\lambda_n} \tan(\lambda_n l) = 0, \qquad \omega_n^2 > \frac{12c_2^2}{h^2}. \tag{4.6}$$

In [9], it is shown that the eigenfunctions satisfy the orthogonality conditions

$$\int_{-l}^{l} [\rho h W_n(x) W_m(x) + \rho I \beta_n(x) \beta_m(x)]\, dx = B_m \delta_{nm},$$

$$\int_{-l}^{l} \Big(EI \frac{\partial \beta_n}{\partial x} \frac{\partial \beta_m}{\partial x} + Gh(W_n' - \beta_n)(W_m' - \beta_m) \Big)\, dx = \omega_m^2 B_m \delta_{nm},$$

$$B_m = \int_{-l}^{l} [\rho h W_m^2(x) + \rho I \beta_m^2(x)]\, dx.$$

We seek a solution of the problem in the form

$$w(x,t) = \sum_{k=1}^{N} A_k(t) W_k(x), \qquad \beta(x,t) = \sum_{k=1}^{N} A_k(t) \beta_k(x).$$

In the expansions for $w(t)$ and $\beta(t)$, the coefficients $A_k(t)$ are the same. We have

$$\frac{\partial w}{\partial t} = \sum_{k=1}^{N} A_k'(t) W_k(x) + l' \sum_{k=1}^{N} A_k(t) \frac{\partial W_k}{\partial l},$$

$$\frac{\partial^2 w}{\partial t^2} = \sum_{k=1}^{N} A_k''(t) W_k(x) + 2l' \sum_{k=1}^{N} A_k'(t) \frac{\partial W_k}{\partial l} + l'' \sum_{k=1}^{N} A_k(t) \frac{\partial W_k}{\partial l} + l'^2 \sum_{k=1}^{N} A_k(t) \frac{\partial^2 W_k}{\partial l^2},$$

$$\frac{\partial \beta}{\partial t} = \sum_{k=1}^{N} A_k'(t) \beta_k(x) + l' \sum_{k=1}^{N} A_k(t) \frac{\partial \beta_k}{\partial l},$$

$$\frac{\partial^2\beta}{\partial t^2} = \sum_{k=1}^{N} A_k''(t)\beta_k(x) + 2l' \sum_{k=1}^{N} A_k'(t)\,\frac{\partial\beta_k}{\partial l} + l'' \sum_{k=1}^{N} A_k(t)\,\frac{\partial\beta_k}{\partial l} + l'^2 \sum_{k=1}^{N} A_k(t)\,\frac{\partial^2\beta_k}{\partial l^2},$$

$$T = \frac{1}{2}\sum_{m=1}^{N} A_m'^2(t)B_m + l' \sum_{k,m=1}^{N} C_{mk}A_k(t)A_m'(t) + \frac{l'^2}{2}\sum_{k,m=1}^{N} D_{mk}A_k(t)A_m(t),$$

$$\Pi = \frac{1}{2}\sum_{m=1}^{N} A_m^2(t)\omega_m^2 B_m.$$

Here

$$C_{mk} = \int_{-l}^{l} \Big(\rho h W_m(x)\,\frac{\partial W_k}{\partial l} + \rho I \beta_m(x)\,\frac{\partial \beta_k}{\partial l}\Big)\,dx,$$

$$D_{mk} = \int_{-l}^{l} \Big(\rho h\,\frac{\partial W_m}{\partial l}\,\frac{\partial W_k}{\partial l} + \rho I\,\frac{\partial \beta_m}{\partial l}\,\frac{\partial \beta_k}{\partial l}\Big)\,dx.$$

We substitute these expressions into the equations of motion for the beam, multiply the first equation by $\beta_m(x)$ and the second by $W_m(x)$, integrate them over x, and combine with each other. As a result, taking into account the orthogonality relations, we obtain

$$[A_m''(t) + \omega_m^2 A_m]B_m + 2l' \sum_{k=1}^{N} C_{mk}A_k'(t) + l'' \sum_{k=1}^{N} C_{mk}A_k(t) + l'^2 \sum_{k=1}^{N} G_{mk}A_k(t) = PU_m, \tag{4.7}$$

where

$$U_m = \int_{-l}^{l} W_m(x)\,dx, \qquad G_{mk} = \int_{-l}^{l} \Big(\rho h W_m(x)\,\frac{\partial^2 W_k}{\partial l^2} + \rho I \beta_m(x)\,\frac{\partial^2 \beta_k}{\partial l^2}\Big)\,dx.$$

The energy balance equation becomes

$$P\sum_{k=1}^{N}\Big(A_k'U_k + l'A_k\,\frac{\partial U_k}{\partial l}\Big) = \sum_{m=1}^{N}(A_m'' + \omega_m^2 A_m)A_m'B_m$$

$$+ \frac{l'}{2}\sum_{m=1}^{N}\Big(A_m'^2\,\frac{\partial B_m}{\partial l} + A_m^2\,\frac{\partial\,(\omega_m^2 B_m)}{\partial l}\Big) + l'' \sum_{k,m=1}^{N} C_{mk}A_kA_m' + l'^2 \sum_{k,m=1}^{N} \frac{\partial C_{mk}}{\partial l}\,A_kA_m'$$

$$+ l' \sum_{k,m=1}^{N} C_{mk}(A_k'A_m' + A_kA_m'') + l'l'' \sum_{k,m=1}^{N} D_{mk}A_kA_m + \frac{l'^3}{2}\sum_{k,m=1}^{N} \frac{\partial D_{mk}}{\partial l}\,A_kA_m$$

$$+ l'^2 \sum_{k,m=1}^{N} D_{mk}A_kA_m' + 4\gamma l'\theta(l'). \tag{4.8}$$

We multiply Eq. (4.7) by A_m' and sum the result over m. Subtracting the resulting equation from (4.8) and using the relation $\partial C_{mk}/\partial l = G_{mk} + D_{mk}$, we have

$$P\sum_{m=1}^{N} A_m\,\frac{\partial U_m}{\partial l} = \frac{1}{2}\sum_{m=1}^{N}\Big(A_m'^2\,\frac{\partial B_m}{\partial l} + A_m^2\,\frac{\partial\,(\omega_m^2 B_m)}{\partial l}\Big) + 2l' \sum_{k,m=1}^{N} D_{mk}A_kA_m'$$

$$+ \sum_{k,m=1}^{N} C_{mk}(A_kA_m'' - A_k'A_m') + l'' \sum_{k,m=1}^{N} D_{mk}A_kA_m + \frac{l'^2}{2}\sum_{k,m=1}^{N} \frac{\partial D_{mk}}{\partial l}\,A_kA_m + 4\gamma\theta(l').$$

Substitution of the expression for A_m'' from (4.7) into the above equation yields the following equation for crack propagation:

$$l'' \sum_{k,m=1}^{N} F_{mk} A_k A_m = P \sum_{k=1}^{N} A_k \Big(\frac{\partial U_k}{\partial l} - \sum_{m=1}^{N} \frac{C_{mk} U_m}{B_m} \Big)$$

$$- \frac{1}{2} \sum_{m=1}^{N} \Big(A_m'^2 \frac{\partial B_m}{\partial l} + A_m^2 \frac{\partial (\omega_m^2 B_m)}{\partial l} \Big) + \sum_{k,m=1}^{N} C_{mk} \omega_m^2 A_k A_m + \sum_{k,m=1}^{N} C_{mk} A_k' A_m'$$

$$- 2l' \sum_{k,m=1}^{N} F_{mk} A_k A_m' - \frac{l'^2}{2} \sum_{k,m=1}^{N} H_{mk} A_k A_m - 4\gamma\theta(l'), \qquad (4.9)$$

$$F_{mk} = D_{mk} - \sum_{j=1}^{N} \frac{C_{jk} C_{jm}}{B_j}, \qquad H_{mk} = \frac{\partial D_{mk}}{\partial l} - 2 \sum_{j=1}^{N} \frac{C_{jk} G_{jm}}{B_j}.$$

The expressions for the coefficients of the matrices are lengthy and are not given here. The derivative $\partial \omega_n^2 / \partial l$ is calculated by differentiating Eqs. (4.5) and (4.6) and is subsequently used to calculate the other derivatives.

The initial conditions (3.2) lead to

$$A_m(0) = A_m'(0) = 0, \qquad l(0) = l_0, \qquad l'(0) = 0. \qquad (4.10)$$

Thus, we have Eqs. (4.7) and (4.9), inequality (1.3), and initial conditions (4.10) to calculate the motion of the beam. This system of second-order nonlinear equations reduces to a system of first-order equations and, together with inequality (1.3), is solved using the fourth-order Runge–Kutta method [6]. As in the case of an Euler beam, for the solution of the system to be unique, it is necessary that the inequality $R = \sum_{j=1}^{N} F_{mk} A_m A_k \geq \varepsilon > 0$ be satisfied. The first time step was calculated for fixed boundaries of the crack. In the next time steps, the quantity $R(t)$ was controlled, and in the calculations performed, it was positive. The calculation results for different number of modes (5, 10, and 19) are in good agreement.

5. Numerical Results. As an example, we consider a wood beam with the following parameters: $E = 10^9$ N/m^2, $\nu = 0.3$, $\rho = 500$ kg/m^3, and $l_0 = 0.3$ m. The surface energy density of the glued layer is set equal to 0.5 N/m. The beam is 0.02 and 0.03 m thick, and the load is varied in the range $P = 500$–900 N/m^2 for a beam 0.02 m thick and in the range $P = 1100$–1500 N/m^2 for a beam 0.03 m thick.

Because the applied load is constant in time, the critical load can be calculated using the theory of equilibrium Barenblatt cracks [10]. A crack propagating in an Euler beam (layering of this beam) is studied in [1, 5]. For an Euler beam, the crack growth condition for the steady-state problem was obtained in [5] using theory [10], and for the unsteady problem, it was obtained in [1] using the least action principle. These conditions are the same. In the case considered, where the beam is single and contains a crack with two ends, this condition becomes

$$M(l,t) = EI \frac{\partial^2 w}{\partial x^2}(l,t) = 2\sqrt{\gamma EI}. \qquad (5.1)$$

In [1], it is noted that, in the beam approximation, condition (5.1) plays the same role as the condition of finiteness of Barenblatt stresses [10].

The solution of the static deflection problem for an Euler beam becomes

$$w(x) = P(x^2 - l_0^2)^2/(24EI).$$

From (5.1) we obtain the crack growth condition

$$w(x) = P(x^2 - l_0^2)^2/(24EI).$$

For a beam 0.02 m thick, we have $P_0 \simeq 1217$ N/m^2. The solution of the unsteady problem with zero initial conditions has the from

$$w(x,t) = \frac{P}{EI} \sum_{k=1}^{\infty} \frac{U_k}{\lambda_k^4 B_k} [1 - \cos(\omega_k t)] W_k(x), \qquad \omega_k^2 = \frac{EI\lambda_k^4}{\rho h},$$

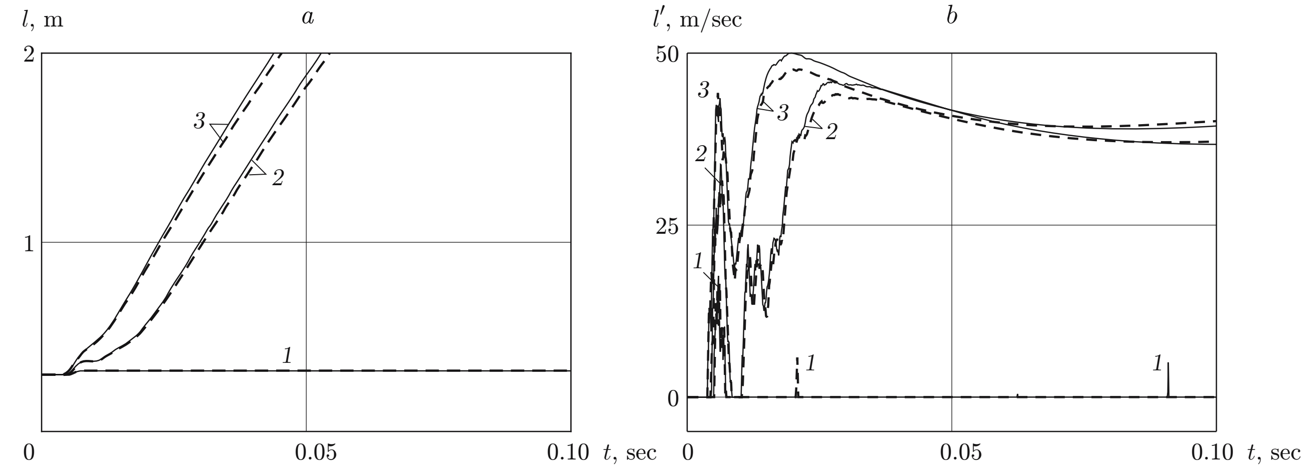

Fig. 2. Curves of $l(t)$ (a) and $l'(t)$ (b) for $P = 700$ (1), 800 (2), and 900 N/m solid curves refer to an Euler beam and dashed curves to a Timoshenko beam.

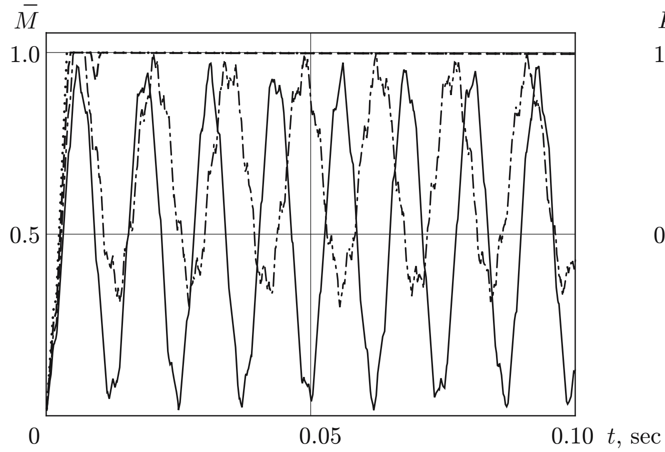

Fig. 3

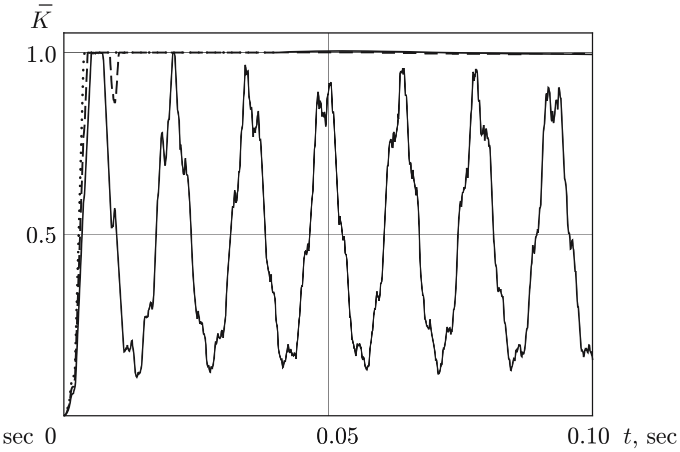

Fig. 4

Fig. 3. Dimensionless momentum $\bar{M}$ versus time for $P = 600$ (solid curve), 700 (dot-and-dashed curve), 800 (dashed curve), and 900 N/m (dotted curve).

Fig. 4. Curves of $\bar{K}(t)$ for $P = 700$ (solid curve), 800 (dashed curve), and 900 N/m (dotted curve).

where U_k and B_k are given by expressions (2.3) and (2.5)–(2.7). From (5.1) we obtain the crack growth condition $P > P_m$. For beams 0.02 and 0.03 m thick, we have $P_m \simeq 600$, and 1103 N/m, respectively.

The condition at the crack end that provides crack growth for a Timoshenko beam was obtained in [3]. In the case considered, this condition is written as

$$K \equiv \Big(\frac{\partial w}{\partial x}\Big)^2 \frac{c_2^2}{a^2}\Big[1 - \frac{1}{c_2^2}\Big(\frac{dl}{dt}\Big)^2\Big] + \Big(\frac{\partial \beta}{\partial x}\Big)^2\Big[1 - \frac{1}{c_0^2}\Big(\frac{dl}{dt}\Big)^2\Big] = \frac{4\gamma}{EI}. \qquad (5.2)$$

Numerical calculations of the unsteady problem using the above algorithm showed the following. For a load $P = 600$ N/m, the crack did not grow, and the motion of the beam corresponded to the solution of the problem with fixed crack ends. For both beam models, and insignificant crack began growth at a load $P = 615$ N/m. At a moderate supercritical load, the crack growth had a stepwise nature: periods of crack growth alternated with periods of its cessation. Having reached a certain length at which the specified load was no longer supercritical, the crack ceased to grow. Thus, at a load $P = 700$ N/m, the crack grew in the case of both Euler and Timoshenko beams. The most significant increase in crack length occurred in a period of 0.01 sec, which was followed by small

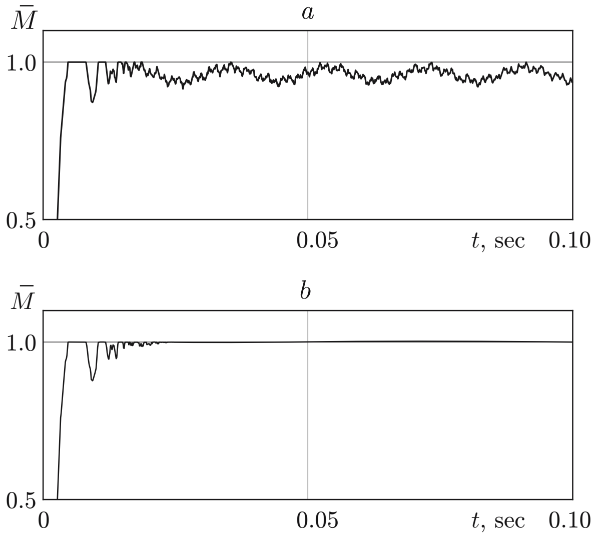

Fig. 5

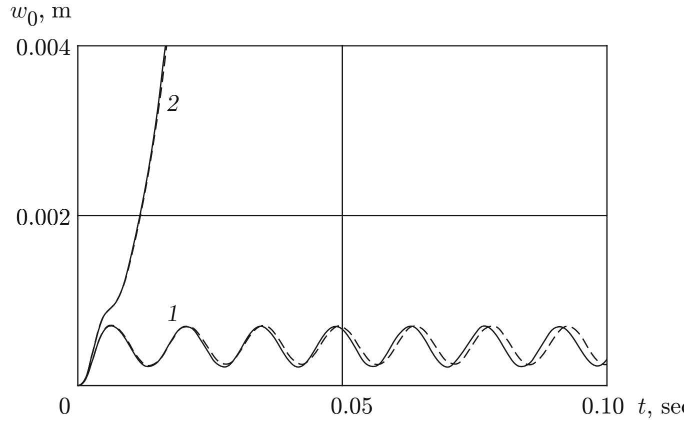

Fig. 6

Fig. 5. Curve of $\bar{M}(t)$ for an Euler beam for $P = 778$ (a) and 779 N/m (b).

Fig. 6. Beam deflection at its middle versus time $[w_0(t) = w(0, t)]$ for $P = 700$ (1) and 900 N/m (2): solid curves refer to an Euler beam; dashed curves to a Timoshenko beam.

periods of growth, during which the crack length increased only slightly. In the case of an Euler beam, the crack growth continued for 0.615 sec, and its final length was 0.321 m. For a Timoshenko beam, the time of crack growth was 0,898 sec, and the final length was 0.322 m.

At a great supercritical load, the calculations predicted rapid unlimited crack growth. However, the linear beam model is suitable only for its small deflections, i.e., in the initial stage of crack growth, and, subsequently, it is necessary to use the nonlinear model. Thus, at a load $P = 800$ N/m for both beam models, the crack length first began to increase and then stopped, after which rapid crack growth began. At a load $P = 900$ N/m, rapid crack growth began at once. The change in the types of solutions corresponding to the limited and unlimited crack growth occurred at a load $P = 779$ N/m for an Euler beam and at $P = 778$ N/m for a Timoshenko beam.

Figure 2 shows curves of $l(t)$ and $l'(t)$ for various loads. It is evident that the solutions for the two (Euler and Timoshenko) beam models are in good agreement with each other. In [3], it is argued that, in the case of an Euler beam, the crack propagation velocity can be arbitrarily high. However, in the example considered and in other calculations, this was not observed. Moreover, for the Euler and Timoshenko beam models, the crack propagation velocities are close and the crack growth process has a stepwise nature.

Figure 3 shows a curve of the flexural momentum at the crack tip $\bar{M}(t) = M(l, t)/(2\sqrt{\gamma EI}\,)$ versus time for an Euler beam, and Fig. 4 shows a curve of the quantity $\bar{K} = KEI/(4\gamma)$ versus time for a Timoshenko beam. From Figs. 2–4 it follows that, during crack growth, conditions (5.1) for an Euler beam and conditions (5.2) for a Timoshenko beam are satisfied with high accuracy although the algorithm for the solution of the problem does not contain these conditions. Figure 5 shows curves of $\bar{M}(t)$ for $P = 778$, 779 N/m, in which one can see the change in the type of solution. Values $\bar{M} = 1$ correspond to the periods of crack growth and values $\bar{M} < 1$ to the periods growth cessation. As the load is increased, the value of $\bar{M}(t)$ approaches unity.

Figure 6 shows a curve of the beam deflection in its middle versus time $w_0(t) = w(0, t)$ for $P = 700$ and 900 N/m. At a load $P = 700$ N/m, which is somewhat larger than the critical load, the motion of the beam has an oscillatory nature. At $P = 900$ N/m — a load far exceeding the critical value — the crack length and the beam deflection increase monotonically.

Similar calculations were performed for $h_0 = 0.03$ m; the corresponding curves for the Euler and Timoshenko beam models differ only slightly. The results lead to the conclusion that the Euler beam model can be used to calculate the strength of lining coatings and beams.

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